THE ERDŐS–KO–RADO THEOREM FOR TWISTED GRASSMANN GRAPHS

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Abstract. We present a "modern" approach to the Erdős–Ko–Rado theorem for Q-polynomial distance-regular graphs and apply it to the twisted Grassmann graphs discovered in 2005 by van Dam and Koolen.

1. Introduction

The 1961 theorem of Erdős, Ko and Rado [8] asserts that the largest possible families Y of d-subsets of a v-set such that $|x \cap y| \geqslant t$ for all $x, y \in Y$ where $v > (t+1)(d-t+1)$ are the families of all d-subsets containing some fixed t-subset. In fact, the exact bound $v > (t+1)(d-t+1)$ was obtained later by Wilson [26] as an application of Delsarte's linear programming method [6]. It is natural to think of this theorem as a result about (vertex) subsets of the Johnson graphs $J(v, d)$, and analogous theorems are known for several other families of distance-regular graphs, e.g., Hamming graphs $H(d, q)$ ($q \geqslant t+2$) [19], Grassmann graphs $J_q(v, d)$ ($v \geqslant 2d$) [14, 10, 11, 23], bilinear forms graphs $\mathrm{Bil}_q(d, e)$ ($d \leqslant e$) [15, 11, 23].

In this note, we first distill common algebraic techniques found in some of the proofs of these "Erdős–Ko–Rado theorems" into a unified approach for general Q-polynomial distance-regular graphs Γ.[1] Our approach is also "modern" in the sense that it is based on and motivated by the theory of two parameters, *width* w and *dual width* w^*, of a subset Y of Γ introduced in 2003 by Brouwer et al. [4]. In this setting, the "t-intersecting" condition amounts to requiring $w \leqslant d-t$ where d is the diameter of Γ, and we shall view the Erdős–Ko–Rado theorem as characterizing those subsets Y with $w = d-t$ and $w^* = t$ by their sizes among all t-intersecting families. There are two steps involved: (1) construction of a specific feasible solution to the dual of a linear programming problem; (2) classification of the *descendents* [24] of Γ, i.e., those subsets having the property $w + w^* = d$. We demonstrate this approach by deriving the Erdős–Ko–Rado theorem for the *twisted Grassmann graphs* $\tilde{J}_q(2d+1, d)$ discovered in 2005 by van Dam and Koolen [5].

2. A "modern" approach to the Erdős–Ko–Rado theorem for Q-polynomial distance-regular graphs

Let $\Gamma = (X, R)$ be a finite connected simple graph with diameter d and path-length distance ∂, and $\mathbb{R}^{X \times X}$ the set of real matrices with rows and columns indexed by X. For each i ($0 \leqslant i \leqslant d$), let $A_i \in \mathbb{R}^{X \times X}$ be the adjacency matrix of

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[1] Q-polynomial distance-regular graphs are thought of as finite/combinatorial analogues of symmetric spaces of rank one; see [2, pp. 311–312].

the distance-i graph Γ_i of Γ, so $A_0 = I$ and $\sum_{i=0}^{d} A_i = J$, the all ones matrix. We say Γ is *distance-regular* if $\boldsymbol{A} := \mathrm{span}\{A_0, A_1, \dots, A_d\}$ is closed under ordinary matrix multiplication; or equivalently, $\boldsymbol{A}$ is a (commutative) algebra. (The reader is referred to [2, 3, 13] for background material on distance-regular graphs.) Throughout this note, suppose Γ is distance-regular. We call $\boldsymbol{A}$ the *Bose–Mesner algebra* of Γ. It is semisimple (as it is closed under transposition) and therefore has a basis $\{E_i\}_{i=0}^{d}$ consisting of the primitive idempotents; we always set $E_0 = |X|^{-1}J$. Note that $\boldsymbol{A}$ is also closed under entrywise multiplication, denoted $\circ$. We shall assume Γ is *Q-polynomial* with respect to the ordering $\{E_i\}_{i=0}^{d}$, i.e., $E_1 \circ E_i$ is a linear combination of E_{i-1}, E_i, E_{i+1} with nonzero coefficients for E_{i-1}, E_{i+1} $(0 \leqslant i \leqslant d)$, where $E_{-1} = E_{d+1} = 0$. Let $Q = (Q_{ij})_{0 \leqslant i,j \leqslant d}$ be the *second eigenmatrix* of Γ:

$$E_j = \frac{1}{|X|} \sum_{i=0}^{d} Q_{ij} A_i \quad (0 \leqslant j \leqslant d).$$

Let Y be a nonempty subset of X and $\chi \in \mathbb{R}^X$ its (column) characteristic vector. Brouwer et al. [4] defined the *width* w and *dual width* w^* of Y as follows:

$$w = \max\{i : \chi^{\mathsf{T}} A_i \chi \neq 0\}, \quad w^* = \max\{i : \chi^{\mathsf{T}} E_i \chi \neq 0\}.$$

They showed (among other results) that

$$w + w^* \geqslant d. \tag{1}$$

We call Y a *descendent* [24] of Γ if $w + w^* = d$. It should be remarked that every descendent is a so-called completely regular code (cf. [17]), and that the induced subgraph is a Q-polynomial distance-regular graph provided it is connected; see [4, Theorems 1–3]. See also [24] for more information on descendents.

Now fix an integer t $(0 < t < d)$ and suppose $w \leqslant d - t$; in other words, Y is "t-intersecting". We recall the inner distribution $\boldsymbol{e} = (e_0, e_1, \dots, e_d)$ of Y:

$$e_i = \frac{1}{|Y|} \chi^{\mathsf{T}} A_i \chi, \quad (\boldsymbol{e}Q)_i = \frac{|X|}{|Y|} \chi^{\mathsf{T}} E_i \chi \quad (0 \leqslant i \leqslant d).$$

It follows that $|Y| = (\boldsymbol{e}Q)_0$ and

$$e_0 = 1, \quad e_1 \geqslant 0, \dots, e_{d-t} \geqslant 0, \quad e_{d-t+1} = \cdots = e_d = 0,$$
$$(\boldsymbol{e}Q)_1 \geqslant 0, \dots, (\boldsymbol{e}Q)_d \geqslant 0.$$

(Observe that the E_i are positive semidefinite.) Following [6], we view these as a linear programming maximization problem. A vector $\boldsymbol{f} = (f_0, f_1, \dots, f_d)$ satisfying (2), (3) below gives a feasible solution to its dual problem:

$$f_0 = 1, \quad f_1 = \cdots = f_t = 0, \quad f_{t+1} > 0, \dots, f_d > 0, \tag{2}$$

$$(\boldsymbol{f}Q^{\mathsf{T}})_1 = \cdots = (\boldsymbol{f}Q^{\mathsf{T}})_{d-t} = 0. \tag{3}$$

Indeed, we have

$$|Y| = (\boldsymbol{e}Q)_0 \leqslant \boldsymbol{e}Q\boldsymbol{f}^{\mathsf{T}} = (\boldsymbol{f}Q^{\mathsf{T}})_0$$

with equality if and only if $(\boldsymbol{e}Q)_{t+1} = \cdots = (\boldsymbol{e}Q)_d = 0$, i.e., $w^* \leqslant t$. By virtue of (1), it follows that

Lemma 1. *Let Y be a nonempty subset of X with $w \leqslant d - t$. Suppose there is a vector $\boldsymbol{f} = (f_0, f_1, \dots, f_d)$ satisfying (2), (3). Then $|Y| \leqslant (\boldsymbol{f}Q^{\mathsf{T}})_0$, and equality holds if and only if Y is a descendent of Γ with $w = d - t$ and $w^* = t$.* ∎

The vector $\boldsymbol{f}$ above is of independent interest from the point of view of *Leonard systems*[2] [25] and will be discussed in detail in a future paper. Here we mention that $\boldsymbol{f}$ can be found for the following graphs:

Γ	$(\boldsymbol{f}Q^{\mathsf{T}})_0$
$J(v,d)$ $(v > (t+1)(d-t+1))$	$\binom{v-t}{d-t}$
$H(d,q)$ $(t = d-1$; or $q \geqslant d$; or $q = d-1,\ t < d-2)$	q^{d-t}
$J_q(v,d)$ $(v \geqslant 2d)$	$\begin{bmatrix} v-t \\ d-t \end{bmatrix}_q$
$\mathrm{Bil}_q(d,e)$ $(d \leqslant e)$	$q^{(d-t)e}$

For $\Gamma = J(v,d)$ or $J_q(v,d)$ (with v,d as in the table), Wilson and Frankl [26, 10] constructed a matrix $B \in \boldsymbol{A}$ such that (i) $B_{xy} = 0$ if $\partial(x,y) \leqslant d-t$; (ii) $B + I - \begin{bmatrix} v-t \\ d-t \end{bmatrix}^{-1} J$ is positive semidefinite and its i^{th} eigenvalue λ_i is positive precisely when $t+1 \leqslant i \leqslant d$, where we interpret $\begin{bmatrix} m \\ n \end{bmatrix}$ as $\binom{m}{n}$ for $J(v,d)$ and $\begin{bmatrix} m \\ n \end{bmatrix}_q$ for $J_q(v,d)$. We define $\boldsymbol{f}$ by $f_0 = 1$, $f_1 = \cdots = f_t = 0$, and $f_i = \begin{bmatrix} v-t \\ d-t \end{bmatrix} \begin{bmatrix} v \\ d \end{bmatrix}^{-1} \lambda_i$ for $t+1 \leqslant i \leqslant d$. For $\Gamma = \mathrm{Bil}_q(d,e)$ $(d \leqslant e)$, Delsarte [7] constructed a *Singleton system*, i.e., a subset whose inner distribution $\boldsymbol{e}' = (e'_0, e'_1, \ldots, e'_d)$ satisfies $e'_1 = \cdots = e'_t = 0$ and $(\boldsymbol{e}'Q)_1 = \cdots = (\boldsymbol{e}'Q)_{d-t} = 0$. It follows that $e'_{t+1}, \ldots, e'_d$ are positive; see [23, §4]. We define $\boldsymbol{f} = \boldsymbol{e}' \cdot \mathrm{diag}(k_0, k_1, \ldots, k_d)^{-1}$ where k_i is the valency of Γ_i $(0 \leqslant i \leqslant d)$. For $\Gamma = H(d,q)$, a subset having the above properties is known as an MDS code [18, Chapter 11]. MDS codes may not exist for some d,q,t, but still $\boldsymbol{e}'$ makes sense and is uniquely determined. If $t = d-1$ or $q \geqslant d$, or if $q = d-1$ and $t < d-2$, then it follows that $e'_{t+1}, \ldots, e'_d$ are positive; see e.g., [9, Appendix]. We again define $\boldsymbol{f} = \boldsymbol{e}' \cdot \mathrm{diag}(k_0, k_1, \ldots, k_d)^{-1}$.

Concerning the conclusion of Lemma 1, the classification of descendents has been done for the 15 known infinite families of Q-polynomial distance-regular graphs with so-called classical parameters and with unbounded diameter, including the above 4 families; see [4, 23, 24]. Moon [19] showed that the upper bound q^{d-t} for $H(d,q)$ and the characterization of its descendents as optimal intersecting families are valid under the (in general) weaker assumption $q \geqslant t+2$. Dual polar graphs discussed in [23] do not always possess $\boldsymbol{f}$ even for the case $t = 1$ [22]; see [21], however, for a description of optimal 1-intersecting families.

3. The Erdős–Ko–Rado theorem for twisted Grassmann graphs

Let q be a prime power and fix a hyperplane H of $\mathbb{F}_q^{2d+1}$. Let X_1 be the set of $(d+1)$-dimensional subspaces of $\mathbb{F}_q^{2d+1}$ not contained in H, and X_2 the set of $(d-1)$-dimensional subspaces of H. The *twisted Grassmann graph* $\Gamma = \tilde{J}_q(2d+1,d)$ [5] has vertex set $X = X_1 \cup X_2$, and two vertices $x, y \in X$ are adjacent if $\dim x + \dim y - 2\dim x \cap y = 2$. It has the same parameters (i.e., the structure constants of $\boldsymbol{A}$) as $J_q(2d+1,d)$. The twisted Grassmann graphs provide the first known family of non-vertex-transitive distance-regular graphs with unbounded diameter. See [12, 1, 20] for more information.

The Erdős–Ko–Rado theorem for $\tilde{J}_q(2d+1,d)$ can now be rapidly obtained. Note that $J_q(2d+1,d)$ and $\tilde{J}_q(2d+1,d)$ share the same Q. Hence we may use the vector $\boldsymbol{f}$ for $J_q(2d+1,d)$ constructed in §2, and Lemma 1 applies. The descendents

[2] Leonard systems provide a linear algebraic framework characterizing the terminating branch of the Askey scheme [16] of (basic) hypergeometric orthogonal polynomials.

of $\tilde{J}_q(2d+1,d)$ have recently been classified by the author [24, Theorem 8.20]. To summarize:

Theorem 2. *Let Y be a nonempty subset of $\tilde{J}_q(2d+1,d)$ with width $w \leqslant d-t$, where $0<t<d$. Then $|Y| \leqslant \left[{2d+1-t \atop d-t}\right]_q$, and equality holds if and only if $Y=\{x \in X_2 : u \subseteq x\}$ for some subspace u of H with $\dim u = t-1$.* ∎

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